For this hw, recall the variation of parameters formula:

Let y_1, y_2 be linearly independent solutions to

$$y'' + a_1(x)\,y' + a_0(x)\,y = 0$$

Then a particular solution to

$$y'' + a_1(x)\,y' + a_0(x)\,y = b(x)$$

is given by

$$y_p = v_1 y_1 + v_2 y_2$$

where

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}\,dx \quad \text{and} \quad v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)}$$

$\boxed{1(a)}$ Solve $y'' - 4y' + 4y = (x+1)e^{2x}$

Step 1: Solve $y'' - 4y' + 4y = 0$

The characteristic equation is $r^2 - 4r + 4 = 0$
which has roots

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)} = \frac{4 \pm \sqrt{0}}{2} = 2$$

repeated real root

Thus,

$$\boxed{y_h = c_1 e^{2x} + c_2 x e^{2x}}$$

We will use $y_1 = e^{2x}$, $y_2 = xe^{2x}$ in the next step.

Step 2: Find y_p for $y'' - 4y' + 4y = \underbrace{(x+1)e^{2x}}_{b(x)}$

We need

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$= (e^{2x})(e^{2x} + 2xe^{2x}) - (xe^{2x})(2e^{2x})$$

$$= e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$= e^{4x}$$

Then,

$$v_1 = \int \frac{-y_2 b(x)}{W(y_1,y_2)}\,dx = \int \frac{-xe^{2x}(x+1)e^{2x}}{e^{4x}}\,dx$$

$$= \int \frac{-x(x+1)\cancel{e^{4x}}}{\cancel{e^{4x}}}\,dx = \int(-x^2-x)\,dx = -\frac{x^3}{3} - \frac{x^2}{2}$$

and

$$v_2 = \int \frac{y_1 b(x)}{W(y_1,y_2)}\,dx = \int \frac{e^{2x}(x+1)e^{2x}}{e^{4x}}\,dx$$

$$= \int \frac{(x+1)\cancel{e^{4x}}}{\cancel{e^{4x}}}\,dx = \int (x+1)\,dx = \frac{x^2}{2} + x$$

Thus,

$$\boxed{y_p = v_1 y_1 + v_2 y_2 = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}}$$

Step 3: The general solution to $y'' - 4y' + 4y = (x+1)e^{2x}$ is

$$\boxed{y = \underbrace{c_1 e^{2x} + c_2 x e^{2x}}_{y_h} + \underbrace{\left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}}_{y_p}}$$

This solution is defined for all x, that is on $I = (-\infty, \infty)$

$\boxed{1(b)}$ Solve $y'' + y = \sin(x)$

Step 1: Solve $y'' + y = 0$.

The characteristic polynomial is $r^2 + 1 = 0$

The roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \frac{\pm\sqrt{-4}}{2} = \frac{\pm\sqrt{4}\sqrt{-1}}{2} = \pm\frac{2}{2}i = \underbrace{\pm i}_{0 \pm i}$$

Thus, $y_1 = e^{0x}\cos(x) = \cos(x)$

$y_2 = e^{0x}\sin(x) = \sin(x)$.

And

$$y_h = c_1 y_1 + c_2 y_2 = c_1 \cos(x) + c_2 \sin(x).$$

Step 2: Find y_p for $y'' + y = \sin(x)$

We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos^2(x) - (-\sin^2(x)) = 1$$

Thus,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{-\sin(x)\sin(x)}{1} dx = -\int \sin^2(x) dx$$

$$= -\int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx = -\frac{1}{2}x + \frac{1}{4}\sin(2x)$$

and

$$v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)}\, dx = \int \frac{\cos(x)\sin(x)}{1}\, dx$$

$$= \int \cos(x)\sin(x)\, dx = \int u\, du = \frac{u^2}{2} = \frac{1}{2}\sin^2(x)$$

$u = \sin(x)$
$du = \cos(x)dx$

So,

$$y_p = v_1 y_1 + v_2 y_2 = \left(-\frac{1}{2}x + \frac{1}{4}\sin(2x)\right)\cos(x) + \frac{1}{2}\sin^2(x)\sin(x)$$

Step 3: The general solution is

$$y = y_h + y_p = c_1\cos(x) + c_2\sin(x) - \frac{1}{2}x\cos(x) + \frac{1}{4}\sin(2x)\cos(x) + \frac{1}{2}\sin^2(x)\sin(x)$$

This solution is defined for all x, ie the interval is $I = (-\infty, \infty)$

$\boxed{1(c)}$ Solve $y'' + y = \sec(x)$

<u>Step 1:</u> Solve $y'' + y = 0$.

The characteristic polynomial is $r^2 + 1 = 0$

The roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \frac{\pm\sqrt{-4}}{2} = \frac{\pm\sqrt{4}\sqrt{-1}}{2} = \pm\frac{2}{2}i = \underbrace{\pm i}_{0 \pm i}$$

Thus, $y_1 = e^{0x}\cos(x) = \cos(x)$

$y_2 = e^{0x}\sin(x) = \sin(x)$.

And

$$y_h = c_1\cos(x) + c_2\sin(x).$$

<u>Step 2:</u> Find y_p for $y'' + y = \sec(x)$

We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

$$= \cos^2(x) - (-\sin^2(x)) = 1$$

Thus,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}dx = \int \frac{-\sin(x)\sec(x)}{1}dx = -\int \frac{\sin(x)}{\cos(x)}dx$$

$$= -\int \tan(x)dx = -\ln|\sec(x)|$$

$$v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{\cos(x)\sec(x)}{1} dx = \int \cos(x) \cdot \frac{1}{\cos(x)} dx$$

$$= \int 1 \, dx = x$$

So,

$$y_p = v_1 y_1 + v_2 y_2 = -\ln|\sec(x)| \cdot \cos(x) + x\sin(x)$$

Step 3: The general solution is

$$\boxed{y = y_h + y_p = c_1\cos(x) + c_2\sin(x) - \ln|\sec(x)| \cdot \cos(x) + x\sin(x)}$$

This would be defined on any interval that $\sec(x)$ is defined on. For example, when $-\frac{\pi}{2} < x < \frac{\pi}{2}$ or $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\boxed{①(d)}$ Solve $y'' - 9y = \frac{9x}{e^{3x}}$

Step 1: Solve $y'' - 9y = 0$

The characteristic equation is $r^2 - 9 = 0$

The roots are $r = \pm 3$.

Let $y_1 = e^{3x}$, $y_2 = e^{-3x}$.

So, $y_h = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{-3x}$.

Step 2: Find y_p for $y'' - 9y = \frac{9x}{e^{3x}}$

We have

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -3\underbrace{e^{3x}e^{-3x}} - 3\underbrace{e^{-3x}e^{3x}}$$

$e^{3x-3x} = e^0 = 1$

$$= -3 - 3 = -6$$

So,

$9xe^{-3x}$

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1,y_2)}\,dx = \int \frac{-e^{-3x} \cdot \frac{9x}{e^{3x}}}{-6}\,dx = \frac{9}{6}\int xe^{-6x}\,dx$$

$$= \frac{9}{6}\left(-\frac{1}{6}xe^{-6x} - \int -\frac{1}{6}e^{-6x}\,dx\right) = -\frac{9}{36}xe^{-6x} + \int \frac{9}{36}e^{-6x}\,dx$$

$\boxed{\begin{array}{ll} u = x & du = dx \\ dv = e^{-6x}dx & v = -\frac{1}{6}e^{-6x} \end{array}}$

$$= -\frac{9}{36}xe^{-6x} - \frac{9}{36}\cdot\frac{1}{6}e^{-6x}$$

$$= -\frac{1}{4}xe^{-6x} - \frac{1}{24}e^{-6x}$$

and

$$v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{e^{3x} \cdot \frac{9x}{e^{3x}}}{W(y_1, y_2)} dx = \int \frac{9x}{-6} dx$$

$$= -\frac{3}{2} \cdot \frac{x^2}{2} = -\frac{3}{4} x^2$$

So,

$$y_p = v_1 y_1 + v_2 y_2 = \left(-\frac{1}{4} x e^{-6x} - \frac{1}{24} e^{-6x}\right) \cdot e^{3x}$$

$$+ \left(-\frac{3}{4} x^2\right) \cdot e^{-3x}$$

$$= -\frac{1}{4} x e^{-3x} - \frac{1}{24} e^{-3x} - \frac{3}{4} x^2 e^{-3x}$$

Step 3: The general solution is

$$y = y_h + y_p = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{4} x e^{-3x} - \frac{1}{24} e^{-3x} - \frac{3}{4} x^2 e^{-3x}$$

this function is defined for all x, so the interval would be $I = (-\infty, \infty)$

$\boxed{1 (e)}$ Solve $y'' + 3y' + 2y = \frac{1}{1+e^x}$

Step 1: Solve $y'' + 3y' + 2y = 0$.

The characteristic equation is $r^2 + 3r + 2 = 0$.

This factors as $(r+2)(r+1) = 0$

So, the roots are $r = -1, -2$.

Let $y_1 = e^{-x}$, $y_2 = e^{-2x}$

Then, $y_h = c_1 y_1 + c_2 y_2 = c_1 e^{-x} + c_2 e^{-2x}$

step 2: Find y_p for $y'' + 3y' + 2y = \frac{1}{1+e^x}$

We have

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = (e^{-x})(-2e^{-2x}) - (-e^{-x})(e^{-2x})$$

$$= -2e^{-3x} + e^{-3x} = -e^{-3x}$$

So,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{-e^{-2x} \cdot \frac{1}{1+e^x}}{-e^{-3x}} dx = \int \frac{e^{3x-2x}}{1+e^x} dx$$

$$= \int \frac{e^x}{1+e^x} dx = \int \frac{1}{1+u} du = \ln|1+u| = \ln|1+e^x|$$

$\boxed{u = 1+e^x \quad du = e^x dx}$ $\boxed{1+e^x > 0}$ $= \ln(1+e^x)$

and

$$v_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)} dx = \int \frac{e^{-x} \cdot \frac{1}{1+e^x}}{-e^{-3x}} dx = -\int \frac{e^{3x-x}}{1+e^x} dx$$

$$\int \frac{-e^{2x}}{1+e^x} dx = \int \left(\frac{e^x}{1+e^x} - e^x \right) dx$$

$$\frac{-e^{2x}}{1+e^x} = \frac{e^x}{1+e^x} - e^x$$

Why?

$$\frac{e^x}{1+e^x} - e^x = \frac{e^x - e^x(1+e^x)}{1+e^x}$$

$$= \frac{e^x - e^x - e^{2x}}{1+e^x}$$

$$= \frac{-e^{2x}}{1+e^x}$$

see v_1 calculation

$$= \ln(1+e^x) - e^x$$

Thus,

$$y_p = v_1 y_1 + v_2 y_2 = \ln(1+e^x) \cdot e^{-x} + (\ln(1+e^x) - e^x) \cdot e^{-2x}$$

$$= e^{-x} \ln(1+e^x) + e^{-2x} \ln(1+e^x) - e^{-x}$$

<u>Step 3:</u> The general solution is

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{-2x} + e^{-x} \ln(1+e^x) + e^{-2x} \ln(1+e^x) - e^{-x}$$

This is defined on $I = (-\infty, \infty)$

$\boxed{\text{1(f)}}$ Solve $y'' + 3y' + 2y = \sin(e^x)$

Step 1: Solve $y'' + 3y' + 2y = 0$

The characteristic equation is $r^2 + 3r + 2 = 0$

This factors as $(r+2)(r+1) = 0$

The roots are $r = -2, -1$

Thus, $y_1 = e^{-2x}$, $y_2 = e^{-x}$

And $y = c_1 y_1 + c_2 y_2 = c_1 e^{-2x} + c_2 e^{-x}$

Step 2: Find y_p for $y'' + 3y' + 2y = \sin(e^x)$

We have

$$W(y_1, y_2) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = (e^{-2x})(-e^{-x}) - (e^{-x})(-2e^{-2x})$$

$$= -e^{-3x} + 2e^{-3x}$$

$$= e^{-3x}$$

So,

$$v_1 = \int \frac{-y_2 \cdot b(x)}{W(y_1, y_2)}\, dx = \int \frac{-e^{-x}\sin(e^x)}{e^{-3x}}\, dx$$

$$= -\int e^{2x} \sin(e^x)\, dx = -\int t \sin(t)\, dt$$

$t = e^x$
$dt = e^x dx$

$\boxed{e^{2x} = (e^x)(e^x) = t\,dt}$

$$= -\left[-t\cos(t) - \int -\cos(t)\,dt \right]$$

$u = t \qquad du = dt$

$dv = \sin(t)\,dt \qquad v = -\cos(t)$

$$= t\cos(t) - \sin(t)$$

$$= e^x\cos(e^x) - \sin(e^x)$$

And

$$v_2 = \int \frac{y_1\, b(x)}{W(y_1, y_2)}\,dx = \int \frac{e^{-2x}\sin(e^x)}{e^{-3x}}\,dx = \int e^x \sin(e^x)\,dx$$

$$= \int t\sin(t)\,dt = -t\cos(t) - \int -\cos(t)\,dt$$

$t = e^x$

$dt = e^x\,dx$

$u = t \qquad du = dt$

$dv = \sin(t)\,dt \qquad v = -\cos(t)$

$$= -t\cos(t) + \sin(t) = -e^x\cos(e^x) + \sin(e^x)$$

So,

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \left(e^x\cos(e^x) - \sin(e^x)\right)e^{-2x}$$

$$+ \left(-e^{-x}\cos(e^x) + \sin(e^x)\right)e^{-x}$$

$$= e^{-x}\cos(e^{x}) - e^{-2x}\sin(e^{x})$$
$$- e^{-2x}\cos(e^{x}) + e^{-x}\sin(e^{x})$$

Step 3: The general solution is

$$y = y_h + y_p = c_1 e^{-2x} + c_2 e^{-x}$$
$$+ e^{-x}\cos(e^{x}) - e^{-2x}\sin(e^{x})$$
$$- e^{-2x}\cos(e^{x}) + e^{-x}\sin(e^{x})$$

This solution is defined on $I = (-\infty, \infty)$